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TECHNICAL NOTES

A VARIATIONAL PRINCIPLE FOR A RADIATION PROBLEM

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NOMENCLATURE

L	fin length
N	fin parameter
T	temperature
T_b	base temperature
X	axial distance
x	dimensionless distance, X/L
y	dimensionless temperature, T/T_b

1. INTRODUCTION

CONSIDER a straight fin of uniform thickness and length L. The fin surface radiates to an environment with zero temperature. If we assume that: (i) thermal properties are independent of temperature, (ii) conduction is 1-dim., (iii) the fin tip is insulated, and (iv) there is no fin to base radiation, then the differential equation that describes the problem becomes

$$y''-Ny^4=0, \quad 0<x<1 \tag{1}$$

subject to the following boundary conditions:

$$y(0)=1, \quad y'(1)=0. \tag{2}$$

In equation (1), y is the dimensionless temperature, x is the dimensionless distance and N is the fin parameter—a quantity related to the emissivity base temperature and the dimensions of the fin. The solution to equations (1) and (2) may be expressed (implicitly) in terms of incomplete beta functions [1]. In some generalized radiating fin problems [2], an explicit dependence of y on x is required. In these cases the problem (1) and (2) is solved approximately.

Our intention in this note is to demonstrate a new variational principle presented in ref. [3] for the problem (1) and (2). First, we will obtain an approximate solution for equations (1) and (2) by using the Ritz procedure. After that a method will be presented, based on the value of the functional, for the estimation of the L_∞ norm of the error of the approximate solution. The error estimate procedure complements one given in ref. [3] and is new.

2. VARIATIONAL PRINCIPLE

The non-linear boundary value problem (1) and (2) belongs to a class of boundary value problems for which a variational principle was developed in ref. [3], we conclude that the functional

$$I=\int_0^1\left[y'^2+\frac{N}{5}y^5+\frac{4}{5N^{1/4}}(y'')^{5/4}\right]\mathrm{d}x-(yy')_0^1 \tag{3}$$

attains a minimum $I(y)=0$ on the exact solution y of equations (1) and (2). Let $Y=y+\delta y$ be an approximate solution that satisfies the boundary conditions (2) identically, i.e.

$$\delta y(0)=\delta y'(1)=0. \tag{4}$$

Expanding equation (3) we have

$$I(Y)=I(y)+\delta I(y,\delta y)+\delta^2 I(\psi,\delta y) \tag{5}$$

where

$$\psi=y+\tau(Y-y), \quad 0\leqslant\tau\leqslant 1. \tag{6}$$

Using the fact [3] that $I(y)=\delta I(y,\delta y)=0$, in equation (5) we have

$$I(Y)=\frac{1}{2}\int_0^1\{A(\psi)\delta y''^2+2\delta y'^2+C(\psi)\delta y^2\}\,\mathrm{d}x \tag{7}$$

where

$$A(\psi)=\frac{1}{4N^{1/4}(\psi'')^{3/4}}, \qquad C(\psi)=4N\psi^3. \tag{8}$$

From equation (7) we want to estimate the L_∞ norm of δy,

$$\|\delta y\|_{L_\infty}=\sup_{x\in(0,1)}|\delta y|. \tag{9}$$

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be constants (depending on Y and $f=\|\delta y\|_{L_\infty}$) such that

$$A(\psi)\geqslant\mathscr{L}_1(Y,f); \quad C(\psi)\geqslant\mathscr{L}_2(Y,f). \tag{10}$$

Using the following inequality [4]:

$$\int_0^1\delta y'^2\,\mathrm{d}x\leqslant\varepsilon\int_0^1\delta y''^2\,\mathrm{d}x+K(\varepsilon)\int_0^1\delta y^2\,\mathrm{d}x \tag{11}$$

where $\varepsilon>0$ and $K(\varepsilon)=1/\varepsilon+12$, and using definitions (10) in equation (7) we get

$$2I\geqslant Sf^2. \tag{12}$$

In equation (12), S is given by

$$S=\left\{\lambda_1^2\left[\mathscr{L}_1-\frac{\mathscr{L}_2\varepsilon}{K(\varepsilon)}\right]+2+\frac{\mathscr{L}_2}{K(\varepsilon)}\right\}, \tag{13}$$

and λ_1^2 is the smallest (first) eigenvalue of the spectral problem [3]

$$\phi_n''+\lambda_n^2\phi_n=0, \tag{14}$$

$$\phi_n(0)=\phi_n'(1)=0. \tag{15}$$

It is easy to see that in the present case that

$$\mathscr{L}_1=\frac{1}{4N^{1/4}[N(Y_{\max}+f)^4+(Y'')_{\max}-N(Y_{\min}-f)^4]^{3/4}} \tag{16}$$

and

$$\mathscr{L}_2=4N(Y_{\min}-f)^3. \tag{17}$$

The estimate on $f=\|\delta y\|_{L_\infty}$ is determined as the positive solution of the inequality (12).

Table 1

N	β	I	$\|\delta y\|_{L_\infty} \leqslant$	$\|\delta y\|_{L_2} \leqslant$
0.5	0.157	0.0169	0.113	0.0533
1	0.236	0.03457	0.172	0.0766
2	0.329	0.329	0.25	0.108

Note that the inequality (12) could be 'optimized' by choosing such a value for $\varepsilon > 0$ that will make the estimate that follows from equation (12) minimal.

For completeness we cite the L_2 error bound derived in [3],

$$\|\delta y\|_{L_2} = \left(\int_0^1 \delta y^2 \,\mathrm{d}x\right)^{1/2} = \left(\frac{1}{A_{\min}\lambda_1^4 + 2\lambda_1^2 + C_{\min}}\right)^{1/2}. \quad (18)$$

To illustrate the procedure, we performed calculations with the trial functions of the form

$$Y = 1 - \beta \sin\frac{\pi}{2}x \quad (19)$$

where β is a constant to be determined. Substituting equation (19) into (3) and minimizing with respect to β the values given in Table 1 are obtained. With these values for β, estimates for errors are calculated from equation (12) (with $\varepsilon = 0.1$) and equation (18).

Finally, we note that the accuracy of the approximate solution and the bounds on errors could be improved by taking more elaborate trial functions.

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A COMPARISON OF APPROXIMATION FORMULAS FOR THE HEAT CONDUCTION EQUATION OF A SPHERE

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NOMENCLATURE

A_i	see equation (12)
Bi	Biot number, hr_s/k
C_i	see equation (6)
Fo	Fourier number, dimensionless time, $\alpha t/r_s^2$
Fo_δ	Fo, when δ approaches 1
f	see equation (9)
h	heat-transfer coefficient [W m^{-2} K^{-1}]
k	thermal conductivity of the pellet [W m^{-1} K^{-1}]
r	dimensionless radial distance, referred to r_s
r_s	radius of spherical particle [m]
t	time [s]
T	dimensionless temperature, referred to T_0
T_g	dimensionless gas temperature, referred to T_0
T_s	dimensionless pellet surface temperature, referred to T_0
$T_{s,\delta}$	T_s, when δ approaches 1
T_0	initial temperature [K]

Greek symbols

α	thermal diffusivity of the pellet [m^2 s^{-1}]
δ	dimensionless penetration depth of the thermal wave, measured from the particle surface, referred to r_s
ϑ	dimensionless surface temperature, $(T_s - T_g)/(1 - T_g)$
θ	dimensionless quantity, $1-(1-Bi)\vartheta$

1. INTRODUCTION

The purpose of the present note is to compare approximate solutions of the heat conduction equation for a sphere found in the literature [1–3] and to give an improved approximation for long heating times. In particular, attention is focused on the surface temperature which plays an important role in many engineering problems, e.g. the convective heat transfer to particles or droplets in two-phase flow. As is well known, for many types of boundary conditions, there exist analytical solutions of the heat conduction equation expressed in terms of infinite series [4]. In more complicated cases numerical exact solutions can be obtained from a number of capable and flexible softwares [5]. However, within the scope of a complex two-phase flow problem which can only be treated numerically [1–3], one cannot employ the solution for single particles. This fact is essentially due to insufficient computer field length and too long computation time. Therefore, the use of approximate solutions is necessary. Moreover, the application of approximate methods is justified in practical engineering, where the general validity of a solution is not as important as a fast and simple treatment of the problem to be solved. For that reason, in the present paper a comparison between the approximation formulas mentioned above [1–3], a solution obtained under the assumption of a spatial uniform pellet temperature, and a numerical exact solution is presented. The validity range of the best approximation formula is extended to larger time intervals. The ODE's obtained from this approximation method are solved analytically for the case of constant heat transfer coefficient and constant gas temperature.

2. GOVERNING EQUATION AND METHOD OF APPROXIMATE SOLUTION

The basic equation is the heat conduction equation for a sphere,

$$\frac{\partial(rT)}{\partial Fo} = \frac{\partial^2(rT)}{\partial r^2} \quad (1)$$